EXERCISE – IV ADVANCED SUBJECTIVE QUESTIONS

1. If $\vec{a} \,\&\, \vec{b}$ are non collinear vectors such that, $\vec{p} = (x + 4y)\vec{a} + (2x + y + 1)\vec{b}$ & $\vec{q} = (y - 2x + 2)\vec{a} + (2x - 3y - 1)\vec{b}$, find x & y such that $3\vec{p} = 2\vec{q}$.

2. (a) Show that the points $\vec{a} - 2\vec{b} + 3\vec{c}\,;\, 2\vec{a} + 3\vec{b} - 4\vec{c} \,\&\, -7\vec{b} + 10\vec{c}$ are collinear.

(b) Prove that the points A = (1, 2, 3), B (3, 4, 7), C(–3, –2, –5) are collinear & find the ratio in which B divides AC.

3. Points X & Y are taken on the sides QR & RS, respectively of a parallelogram PQRS, so that $\overrightarrow{QX} = 4\overrightarrow{XR} \,\&\, \overrightarrow{RY} = 4\overrightarrow{YS}$. The line XY cuts the line PR at Z. Prove that $\overrightarrow{PZ} = \left(\frac{21}{25}\right)\overrightarrow{PR}$.

4. Find out whether the following pairs of lines are parallel, non parallel; & intersecting, or non-parallel & non-intersecting.

(i) $\begin{aligned}\vec{r}_1 &= \hat{i} + \hat{j} + 2\hat{k} + \lambda(3\hat{i} - 2\hat{j} + 4\hat{k})\\ \vec{r}_2 &= 2\hat{i} + \hat{j} + 3\hat{k} + \mu(-6\hat{i} + 4\hat{j} - 8\hat{k})\end{aligned}$

(ii) $\begin{aligned}\vec{r}_1 &= \hat{i} - \hat{j} + 3\hat{k} + \lambda(\hat{i} - \hat{j} + \hat{k})\\ \vec{r}_2 &= 2\hat{i} + 4\hat{j} + 6\hat{k} + \mu(2\hat{i} + \hat{j} + 3\hat{k})\end{aligned}$

(iii) $\begin{aligned}\vec{r}_1 &= \hat{i} + \hat{k} + \lambda(\hat{i} + 3\hat{j} + 4\hat{k})\\ \vec{r}_2 &= 2\hat{i} + 3\hat{j} + \mu(4\hat{i} - \hat{j} + \hat{k})\end{aligned}$

5. Let OACB be parallelogram with O at the origin & OC a diagonal. Let D be the mid point of OA. Using vector method prove that BD & CO intersect in the same ratio. Determine this ratio.

6. In $\triangle$ABC, points E and F divide sides AC and AB respectively so that $\frac{AE}{EC} = 4$ and $\frac{AF}{FB} = 1$. Suppose D is a point on side BC. Let G be the intersection of EF and AD and suppose D is situated that $\frac{AG}{GD} = \frac{3}{2}$. If the ratio $\frac{BD}{DC} = \frac{a}{b}$, where a and b are in their lowest form, find the value of (a + b).

7. The resultant of two vectors $\vec{a} \,\&\, \vec{b}$ is perpendicular to $\vec{a}$. If $|\vec{b}| = \sqrt{2}\,|\vec{a}|$ show that the resultant of $2\vec{a} \,\&\, \vec{b}$ is perpendicular to $\vec{b}$.

8. Use vectors to prove that the diagonals of a trapezium having equal non parallel sides are equal & conversely.

9. Given three points on the xy plane on O(0, 0), A(1, 0) and B(–1, 0). Point P is moving on the plane satisfying the condition $(\overrightarrow{PA}.\overrightarrow{PB}) + 3(\overrightarrow{OA}.\overrightarrow{OB}) = 0$.

If the maximum and minimum values of $|\overrightarrow{PA}|\,|\overrightarrow{PB}|$ are M and m respectively then find the value of $M^2 + m^2$.

10. In the plane of triangle ABC, squres ACXY, BCWZ are described, in the order given, externally to the triangle on AC & BC respectively. Given that $\overrightarrow{CX} = \vec{b}, \overrightarrow{CA} = \vec{a}, \overrightarrow{CW} = \vec{x}, \overrightarrow{CB} = \vec{y}$. Prove that $\vec{a}.\vec{y} + \vec{x}.\vec{b} = 0$.

Deduce that $\overrightarrow{AW}.\overrightarrow{BX} = 0$.

11. A $\triangle$OAB is right angled at O ; squares OALM & OBPQ are constructed on the sides OA and OB externally. Show that the lines AP & BL intersect on the altitude through 'O'.

12. Given that

$\vec{u} = \hat{i} - 2\hat{j} + 3\hat{k}\,;\, \vec{v} = 2\hat{i} + \hat{j} + 4\hat{k}\,;\, \vec{w} = \hat{i} + 3\hat{j} + 3\hat{k}$ and

$(\vec{u}.\vec{R} - 10)\hat{i} + (\vec{v}.\vec{R} - 20)\hat{j} + (\vec{w}.\vec{R} - 20)\hat{k} = 0$.

Find the unknown vector $\vec{R}$.

13. The length of the edge of the regular tetrahedron DABC is 'a'. Point E and F are taken on the edges AD and BD respectively such that E divides $\overrightarrow{DA}$ and F divides $\overrightarrow{BD}$ in the ratio 2 : 1 each. Then find the area of triangle CEF.

14. $A(\vec{a})\,;B(\vec{b})\,;C(\vec{c})$ are the vertices of the triangle ABC such that $\vec{a} = \frac{1}{2}(2\hat{i}-\hat{j}-7\hat{k})\,;\vec{b}=3\hat{i}+\hat{j}-4\hat{k}$; $\vec{c}=22\hat{i}-11\hat{j}-9\vec{r}$. A vector $\vec{p}=2\hat{j}-\hat{k}$ is such that $(\vec{r}+\vec{p})$ is parallel to $\hat{j}$ and $(r-2\hat{i})$ is parallel to $\vec{p}$. Show that there exists a point D($\vec{d}$) on the line AB with $\vec{d}=2t\hat{i}+(1-2t)\hat{j}+(1-4)\hat{k}$. Also find the shortest distance C from AB.

15. The position vectors of the points A, B, C are respectively (1, 1, 1) ; (1, −1, 2) ; (0, 2, −1). Find a unit vector parallel to the plane determined by ABC perpendicular to the vector (1, 0, 1).

16. Let $\begin{vmatrix} (a_1-a)^2 & (a_1-b)^2 & (a_1-c)^2 \\ (b_1-a)^2 & (b_1-b)^2 & (b_1-c)^2 \\ (c_1-a)^2 & (c_1-b)^2 & (c_1-c)^2 \end{vmatrix}$ = 0 and if the vectors $\vec{\alpha}=\hat{i}+a\hat{j}+a^2\hat{k}\,;\vec{\beta}=\hat{i}+b\hat{j}+b^2\hat{k}$; $\vec{\gamma}=\hat{i}+c\hat{j}+c^2\hat{k}$ are non coplanar, show that the vectors $\vec{\alpha}_1=\hat{i}+a_1\hat{j}+a_1^2\hat{k}\,;\vec{\beta}_1=\hat{i}+b_1\hat{j}+b_1^2\hat{k}$ and $\vec{\gamma}_1=\hat{i}+c_1\hat{j}+c_1^2\hat{k}$ are coplanar.

17. The pv's of the four angular points of a tetrahedron are : $A(\hat{j}+2\hat{k})\,;B(3\hat{i}+\hat{k})\,;C(4\hat{i}+3\hat{j}+6\hat{k})$ & $D(2\hat{i}+3\hat{j}+2\hat{k})$. Find :

(i) The perpendicular distance from A to the line BC.

(ii) The volume of the tetrahedron ABCD.

(iii) The perpendicular distance from D to the plane ABC.

(iv) The shortest distance between the lines AB & CD.

18. The length of an edge of a cube $ABCDA_1B_1C_1D_1$ is equal to unity. A point E taken on the edge $\overrightarrow{AA_1}$ is such that $|\overrightarrow{AE}|$ = 1/3. A point F is taken on the edge $\overrightarrow{BC}$ such that $|\overrightarrow{BF}|$ = 1/4. If O_1 is the centre of the cube, find the shortest distance of the vertex B_1 from the plane of the ΔO_1EF.

19. The vector $\overrightarrow{OP}=\hat{i}+2\hat{j}+2\hat{k}$ turns through a right angle, passing through the positive x-axis on the way. Find the vector in its new position.

20. Find the point R in which the line AB cuts the plane CDE where $\vec{a}=\hat{i}+2\hat{j}+\hat{k},\ \vec{b}=2\hat{i}+\hat{j}+2\hat{k},$ $\vec{c}=-4\hat{j}+4\hat{k},\ \vec{d}=2\hat{i}-2\hat{j}+2\hat{k}$ & $\vec{e}=4\hat{i}+\hat{j}+2\hat{k}$.

21. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ then show that the value of the scalar triple product $[\,n\vec{a}+\vec{b},\ n\vec{b}+\vec{c},\ n\vec{c}+\vec{a}\,]$ is (n^3 + 1)

$$\begin{vmatrix} \vec{a}.\hat{i} & \vec{a}.\hat{j} & \vec{a}.\hat{k} \\ \vec{b}.\hat{i} & \vec{b}.\hat{j} & \vec{b}.\hat{k} \\ \vec{c}.\hat{i} & \vec{c}.\hat{j} & \vec{c}.\hat{k} \end{vmatrix}$$

22. (A) Prove that $|\vec{a}\times\vec{b}|$ = $\sqrt{-\vec{b}.[\vec{a}\times(\vec{a}\times\vec{b})]}$

(B) Given that $\vec{a},\vec{b},\vec{p},\vec{q}$ are four vectors such that $\vec{a}+\vec{b}=\mu\vec{p},\ \vec{b}.\vec{q}=0$ & $(\vec{b})^2$ = 1, where μ is a scalar then prove that $|(\vec{a}.\vec{q})\vec{p}-(\vec{p}.\vec{q})\vec{a}|=|\vec{p}.\vec{q}|$

23. Find the scalars α & β if $\vec{a}\times(\vec{b}\times\vec{c})+(\vec{a}.\vec{b})\vec{b}$ $=(4-2\beta-\sin\alpha)\vec{b}+(\beta^2-1)\vec{c}$ & $(\vec{c}.\vec{c})\vec{a}=\vec{c}$ while $\vec{b}$ & $\vec{c}$ are non zero non collinear vectors.

24. ABCD is a tetrahedron with pv's of its angular points as A(−5, 22, 5); B(1, 2, 3); C(4, 3, 2) and D(−1, 2, −3). If the area of the triangle AEF where the quadrilaterals ABDE and ABCF are parallelograms is $\sqrt{S}$ then find the value of S.

25. If $\vec{A},\vec{B}$ & $\vec{C}$ are vectors such that $|\vec{B}|$ = $|\vec{C}|$, prove that : $[(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})]\times(\vec{B}\times\vec{C}).(\vec{B}+\vec{C})=0$.

26. Let $\vec{a}=\alpha\hat{i}+2\hat{j}-3\hat{k},\ \vec{b}=\hat{i}+2\alpha\hat{j}-2\hat{k}$ & $\vec{c}=2\hat{i}-\alpha\hat{j}+\hat{k}$. Find the value(s) of α, if any, such that $(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})=0$. Find the vector product when $\alpha = 0$

27. Find a vector $\vec{v}$ which is coplanar with the vectors $\hat{i}+\hat{j}-2\hat{k}$ & $\hat{i}-2\hat{j}+\hat{k}$ and is orthogonal to the vector $-2\hat{i}+\hat{j}+\hat{k}$. It is given that the projection of $\vec{v}$ along the vector $\hat{i}-\hat{j}+\hat{k}$ equal to $6\sqrt{3}$.

28. Consider the non zero vectors $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ such that no three of which are coplanar then prove that $\vec{a}[\vec{b}\vec{c}\vec{d}] + \vec{c}[\vec{a}\vec{b}\vec{d}] = \vec{b}[\vec{a}\vec{c}\vec{d}] + \vec{d}[\vec{a}\vec{b}\vec{c}]$. Hence prove that $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ represent the position vectors of the vertices of a plane quadrilateral if $\dfrac{[\vec{b}\,\vec{c}\,\vec{d}] + [\vec{a}\,\vec{b}\,\vec{d}]}{[\vec{a}\,\vec{c}\,\vec{d}] + [\vec{a}\,\vec{b}\,\vec{c}]} = 1$.

29. The base vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are given in terms of base vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$ as $a_1 = 2\vec{b}_1 + 3\vec{b}_2 - \vec{b}_3$; $\vec{a}_2 = \vec{b}_1 - 2\vec{b}_2 + 2\vec{b}_3$ & $\vec{a}_3 = 2\vec{b}_1 + \vec{b}_2 - 2\vec{b}_3$. If $\vec{F} = 3\vec{b}_1 - \vec{b}_2 + 2\vec{b}_3$, then express $\vec{F}$ in terms of $\vec{a}_1, \vec{a}_2$ & $\vec{a}_3$.

30. If $A(\vec{a})\,;B(\vec{b})\,;C(\vec{c})$ are three non collinear points, then for any point P($\vec{p}$) in the plane of the ΔABC, prove that ;

(i) $[\,\vec{a}\vec{b}\vec{c}\,] = \vec{p}.(\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a})$

(ii) The vector $\vec{v}$ prependicular to the plane of the triangle ABC drawn from the origin 'O' is given by $\vec{v} = \pm \dfrac{[\vec{a}\vec{b}\vec{c}](\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a})}{4\Delta^2}$ where $\vec{\Delta}$ is the vector area of the triangle ABC.

31. (a) If $p\vec{x} + (\vec{x}\times\vec{a}) = \vec{b}\,;(p \neq 0)$ prove that

$$\vec{x} = \frac{p^2\vec{b}(\vec{b}.\vec{a})\vec{a} - p(\vec{b}\times\vec{a})}{p(p^2 + a^2)}$$

(b) Solve the following equation for the vector $\vec{p}\,;\vec{p}\times\vec{a} + (\vec{p}.\vec{b})\vec{c} = \vec{b}\times\vec{c}$ where $\vec{a}, \vec{b}, \vec{c}$ are non zero non coplanar vectors and $\vec{a}$ is neither perpendicular to $\vec{b}$ nor to $\vec{c}$, hence show that $\left(\vec{p}\times\vec{a} + \dfrac{[\vec{a}\vec{b}\vec{c}]}{\vec{a}.\vec{c}}\vec{c}\right)$ is perpendicular to $\vec{b} - \vec{c}$.